

Optimal control models of Einstein's field equations

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ABSTRACT

It is shown that some problems of Einstein's field equations of the theory of General Relativity can be modeled as optimal control problems. The advantages of adopting such an approach are explained. As a demonstration of this interdisciplinary research, four examples are reviewed for problems in relativistic astrophysics and cosmology. The first example, in relativistic astrophysics, uses optimal control to prove that the Tolman–Oppenheimer–Volkoff equation of hydrostatic equilibrium is a necessary condition to extremize the mass of a stellar model. The second example, also in relativistic astrophysics, uses optimal control to estimate an upper limit to the mass of a neutron star. The third example, in relativistic cosmology, uses optimal control to construct a closed universe with maximum lifetime. The fourth example, in cosmological inflation, uses optimal control to construct a model of a slow-roll inflationary universe with minimum change of the scalar field. Results show that optimal control is more powerful than classical variational calculus and that optimal control models of Einstein's field equations add physical significance to their solutions. Extension to other problems is explored, and difficulties in the formulation of optimal control problems in General Relativity are indicated.

Keywords: General Relativity; Stellar Models; Cosmological Models; Optimal Control.

1. Introduction

General Relativity (GR) is Einstein's theory of gravitation. In GR, the interaction between the mass-energy distribution and the spacetime geometry is described by Einstein's field equations. The theory of optimal control deals with optimal control problems, which require control functions to be optimal with respect to given criteria. The theory has many interesting applications in various fields of engineering, economics, management, and biology. However, quite a few applications have been made to GR. This paper presents a review of some problems in GR that have been formulated and solved as optimal control problems. The example problems demonstrate the advantages of adopting such an approach and motivate further applications of optimal control to Einstein's field equations.

In particular, Einstein's interior field equations are potential candidates for optimal control application. It is well known that the interior field equations are under-determined, in the sense that the number of equations is less than the number of variables. To obtain an interior solution, an additional equation has to be introduced. Ideally, an equation of state is to be specified. However, this approach has led to quite a few exact solutions with simple equations of state. When realistic equations of state are specified, the problem becomes analytically formidable, and numerical methods have to be used. More often, a relation between some variables is assumed so that the problem is simplified. In fact, most interior solutions have been obtained this way. An alternative approach is to formulate the problem as an optimal control model. In such a model, state functions are governed by differential equations that include some unknown control functions. The problem is initially under-determined. But the model also incorporates a functional, of the state and control functions, that is required to be optimal. Then, the conditions of optimality determine the control functions and the problem becomes determined. This approach could lead to new physical interpretation and a deeper understanding of the construction of interior field equations and their solutions.

The paper is organized as follows. A brief outline of optimal control models is given in section 2. In section 3, two applications of optimal control to stellar models are reviewed. Sections 4 and 5 give a review of an application, respectively, to cosmology and inflation. In the

concluding section, extension to other problems is suggested, and difficulties in the formulation of optimal control problems in GR are indicated.

2. Optimal Control Models

In such models, a system is defined in terms of an n-dimensional state vector $x(t)$ and an m-dimensional control vector $u(t)$. An optimal control problem [1-3] is to determine the control vector function u which maximizes the objective function

$$J = \int_{t_0}^{t_1} L(x, u, t) dt + F(x_1, t_1), \quad (1)$$

where the system evolution equation is

$$\dot{x} = f(x, u, t), \quad (2)$$

subject to the constraints

$$x(t_0) = x_0, \quad \psi(x_1, t_1) = 0 \quad u \in \Omega. \quad (3)$$

The functions L and F are scalars. The terminal constraint function ψ is an s-dimensional vector. The set Ω is the set of admissible controls. The initial time t_0 is given explicitly but the terminal time t_1 may be unspecified.

The Hamiltonian for this problem is

$$H(x, u, \lambda, t) = L(x, u, t) + \langle \lambda, f(x, u, t) \rangle \quad (4)$$

where λ denotes an n-dimensional vector of costate functions.

According to Pontryagin's maximum principle, the following necessary conditions hold along an optimal trajectory.

$$\bar{u} = \arg \min H(\bar{x}, u, \lambda, t), \quad (5)$$

$$\dot{\lambda} = -H_{,x}(\bar{x}, \bar{u}, \lambda, t), \quad (6)$$

$$\lambda(t_1) = F_{,x_1} + \langle \psi_{,x_1}, \nu \rangle, \quad (7)$$

$$H(t_1) = -F_{,t_1} - \langle \psi_{,t_1}, \nu \rangle, \quad (8)$$

where a comma denotes a partial derivative, \bar{x} and \bar{u} denote the candidate state and control vectors respectively, and ν is an s-dimensional vector of constant Lagrange multipliers associated with ψ .

The terminal conditions (7) and (8) result from the transversality condition

$$[(F_{,x_1} - \lambda(t_1)) \delta x + (H(t_1) + F_{,t_1}) \delta t]_{\psi(x_1, t_1)} = 0. \quad (9)$$

If some initial parameters are not specified similar conditions may be obtained at the initial time t_0 .

The optimal control problem in the above form is referred to as a problem of Bolza. If $F(x_1, t_1) = 0$ then the problem is a problem of Lagrange, while if $L(x, u, t) = 0$ then it is a problem of Mayer. The three problems are equivalent: each of them can be converted to the other by suitably defining the state functions.

3. Application of Optimal Control to Stellar Models

One of the main applications of GR is the construction of stellar models [4]. The simplest such model is an isolated static perfect fluid sphere, of coordinate radius R and effective mass-energy M . The vacuum outside the sphere has the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad r \geq R. \quad (10)$$

In the interior of the sphere, the spacetime is described by the static spherically symmetric metric

$$ds^2 = e^{2\nu} dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad r \leq R, \quad (11)$$

where ν and m are functions of the radial coordinate r . The energy-momentum tensor for a perfect fluid is given by

$$T^{ab} = (\mu + p)U^a U^b - pg^{ab}, \quad (12)$$

where μ and p are, respectively, the fluid density and pressure, and U^a is the four-velocity of the fluid. The field equations reduce to

$$m' = 4\pi r^2 \mu, \quad (13)$$

$$p' = -\frac{(\mu + p)(m + 4\pi r^2 p)}{r(r - 2m)}, \quad (14)$$

$$\nu' = -\frac{2p'}{(\mu + p)}, \quad (15)$$

where a prime denotes differentiation with respect to r .

Now, we give a review of two problems in stellar models, that have been solved using optimal control: a derivation of the equation of hydrostatic equilibrium, and a calculation of the maximum mass of a neutron star.

3.1. The equation of hydrostatic equilibrium

The first law of thermodynamics relates the number density of baryons n to μ and p by the equation

$$\frac{dn}{d\mu} = -\frac{n}{p + \mu}. \quad (16)$$

The following theorem has been proved [5,6].

Theorem 1. Among all momentarily static and spherically symmetric configurations of cold, catalyzed matter that contain a specific number of baryons inside a sphere of radius R ,

$$N = \int_0^R 4\pi r^2 \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} n(r) dr,$$

that configuration which extremizes the mass as sensed from outside,

$$M = \int_0^R 4\pi r^2 \mu(r) dr,$$

satisfies the Tolman-Oppenheimer-Volkoff (TOV) equation of hydrostatic equilibrium [7,8]

$$p' = -\frac{(p + \mu)(m + 4\pi r^3 p)}{r(r - m)}, \quad (17)$$

Proofs to the theorem have been given in [5,6], using classical variational calculus, with each of them filling about two pages. Rather than using classical variational calculus, [9] gives a proof using optimal control. The problem is formulated as an optimal control problem as follows. Determine the control function $n(r)$ which extremizes the objective function

$$M = m(R) = \int_0^R 4\pi r^2 \mu(n) dr, \quad (18)$$

where the system equations and constraints are

$$m' = 4\pi r^2 \mu(n), \quad (19)$$

$$z' = 4\pi r^2 \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} n, \quad (20)$$

$$m(0) = 0, \quad z(0) = 0, \quad (21)$$

$$z(R) = N. \quad (22)$$

Application of Pontryagin's maximum principle led directly to the TOV equation, in a way that is simpler and more elegant than the classical variational calculus. Details are given in [9].

3.2. The maximum mass of a neutron star

In observational astrophysics, an important problem is to identify an object as a black hole or a neutron star. The mass of a stable neutron star should not exceed a certain limit; otherwise, the star collapses into a black hole. The identification of an object depends on whether the object's observed mass exceeds that limit. Hence, an estimate of the maximum mass of a stable neutron star is needed. However, the calculation of the maximum mass depends on the equation of state (EOS), which can be predicted only at certain density ranges. For neutron matter with higher densities, EOS cannot be predicted. By imposing only some physical constraints at higher densities, [8] suggested that the mass of a stable neutron star becomes maximum for the stiffest possible EOS.

By using optimal control, [10] successfully proved this suggestion; namely that, in the regions where it is uncertain, EOS that gives the maximum mass is $p = \mu$.

With a few simplifications, their work can be summarized as follows. Consider a neutron star with field equations (13-15) for a static perfect fluid sphere. Let the star be composed of two regions:

(a) Region A: an inner core with higher densities, where

$$0 \leq r \leq r_0, \quad \mu_c \geq \mu \geq \mu_0, \quad p_c \geq p \geq p_0. \quad (23)$$

(b) Region B: an outer envelope with given EOS $p = p(\mu)$, where

$$r_0 \leq r \leq R, \quad \mu_0 \geq \mu \geq \mu_R, \quad p_0 \geq p \geq 0. \quad (24)$$

The mass contained in B depends not only on the given EOS, but also on the values at the boundary $r = r_0$, and hence it is denoted $M_0(\mu_0, r_0, m_0)$. For the dynamical system, take μ as the independent variable, the functions p, r, m as state functions, and $u = \dot{p}$ as the control function. Then, the optimal control problem is formulated as follows. Determine the control function $u(\mu)$ which maximizes the objective function

$$M = \int_{\mu_c}^{\mu_0} 4\pi r^2 \dot{r} d\mu + M_0(\mu_0, r_0, m_0), \quad (25)$$

subject to the system equations and constraints

$$\dot{p} = u, \quad p(\mu_c) = p_c, \quad (26)$$

$$\dot{r} = -\frac{r(r-m)u}{(p+\mu)(m+4\pi r^3 p)}, \quad r(\mu_c) = 0, \quad (27)$$

$$\dot{m} = -\frac{4\pi\mu r^3(r-m)u}{(p+\mu)(m+4\pi r^3 p)}, \quad m(\mu_c) = 0, \quad (28)$$

$$0 \leq u \leq 1, \quad (29)$$

where a dot denotes differentiation with respect to μ .

Application of Pontryagin's maximum principle leads to the optimal control $u = \dot{p} = 1 \Rightarrow p = \mu$ with possibly a switch to $u = \dot{p} = 0 \Rightarrow p = p_c$. Further calculations result in a maximum mass of about $3.2M_\odot$ for $\mu_0 = 4.6 \times 10^{14} \text{ gcm}^{-3}$. Details are given in [10].

4. Application of Optimal Control to Cosmological Models

Another important application of GR is the construction of cosmological models. The simplest such model is a homogeneous and isotropic spacetime, filled with perfect fluid, with the Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = dt^2 - a^2(t) \left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \right], \quad (30)$$

where a is a function of the cosmic time t , and $k = -1, 0, +1$ corresponds to an open, flat, and closed universe respectively. The function $a(t)$ is called the *scale factor*.

Einstein's field equations, with a *cosmological constant* Λ , may be reduced to

$$3H^2 + \frac{3k}{a^2} = 8\pi\mu + \Lambda, \quad (31)$$

$$2\dot{H} + 3H^2 + \frac{k}{a^2} = -8\pi p + \Lambda, \quad (32)$$

where a dot denotes differentiation with respect to t , and H is the Hubble parameter

$$H = \frac{\dot{a}}{a} \quad (33)$$

Many models consider Einstein’s field equations with a varying cosmological constant and a linear equation of state

$$p = \alpha\mu, \quad \alpha = \text{const.}, \quad 0 \leq \alpha \leq 1, \quad (34)$$

in which case equations (31) and (32) give

$$2\ddot{H} + 6H\dot{H}(1 + \alpha) - \frac{2k}{a^2}H(1 + 3\alpha) = \dot{\Lambda}(1 + \alpha). \quad (35)$$

Then, such models have two equations (33) and (35) in three unknowns $a(t)$, $H(t)$ and $\Lambda(t)$. In order to obtain solutions, authors assumed various forms of Λ , for example [11]-[19]. Rather than using an arbitrary form of Λ , optimal control has successfully been used [20] to construct a model that Λ is determined by the requirement of optimality.

A closed universe with a maximum lifetime [20]

For a closed universe, ($k = +1$), the scale factor increases from the initial size until $\dot{a} = 0$ then contracts. A closed universe has a finite lifetime. It is interesting to design a closed universe with a maximum lifetime. Equations (33) and (35), with $k = +1$, take the form

$$\dot{a} = aH, \quad (36)$$

$$\dot{\Lambda} = \frac{2}{(1 + \alpha)}\ddot{H} + 6H\dot{H} - \frac{2}{a^2}\left(\frac{1 + 3\alpha}{1 + \alpha}\right)H, \quad (37)$$

which is a system of two equations in three unknowns $a(t)$, $H(t)$ and $\Lambda(t)$. Using equations (32) - (34), one obtains

$$\Lambda = \frac{1}{1 + \alpha}\left[2\dot{H} + 3H^2(1 + \alpha) + \frac{1}{a^2}(1 + 3\alpha)\right], \quad (38)$$

$$\mu = \frac{1}{8\pi}\left[3H^2 + \frac{3}{a^2} - \Lambda\right], \quad p = \alpha\mu. \quad (39)$$

For the dynamical system (36) and (37), the functions a , H , $w = \dot{H}$, Λ are taken as state functions, and $u = \dot{w}$ as the control function.

Further, boundary conditions need to be specified. In order to avoid the Big Bang singularity, the initial time, $t = t_i = 0$, is taken shortly after the Big Bang moment, when the size of the universe becomes little greater than zero, namely $a(0) = a_0 > 0$. Assuming $H(0) = 0$, an expanding

universe requires $\dot{H}(0) = A > 0$. On the other hand, the terminal time $t = t_f = T$ is taken when expansion stops, and the size of the universe reaches its maximum, namely $H(T) = 0$ and $\dot{H}(T) < 0$.

Then, the optimal control problem is to determine the control function $u(t)$ which maximizes the objective function

$$J = \int_0^T dt, \tag{40}$$

subject to

$$\dot{a} = aH, \quad a(0) = a_0, \tag{41}$$

$$\dot{H} = w, \quad H(0) = H(T) = 0, \tag{42}$$

$$\dot{w} = u, \quad w(0) = A > 0, \tag{43}$$

$$\dot{\Lambda} = 6Hw - \frac{2}{a^2} \left(\frac{1+3\alpha}{1+\alpha} \right) H + \frac{2}{(1+\alpha)} u, \quad \Lambda(0) = \Lambda_0, \tag{44}$$

$$|u| \leq 1. \tag{45}$$

Application of Pontryagin's maximum principle leads to the optimal solution

$$u = -1, \tag{46}$$

$$H = -\frac{1}{2}t^2 + At, \tag{47}$$

$$a(t) = a_0 \exp\left[\frac{-1}{6}t^2(t - 3A)\right]. \tag{48}$$

The maximum lifetime is

$$2T = 4A. \tag{49}$$

The maximum value of the scale factor is

$$a_{\max} = a(T) = a_0 \exp\left[\frac{2}{3}A^3\right]. \tag{50}$$

This universe expands to the above maximum size at $t = T$ and then contracts to

$$a_f = a(2T) = a_0 \exp\left[-\frac{8}{3}A^3\right]. \tag{51}$$

5. Application of optimal control to inflationary cosmological models

Inflationary models are based on the effective theory of a self-acting scalar field ϕ with self-action potential $V(\phi)$, which interacts minimally with the gravitational field [21]. For such models, with the FRW metric, the field equations, with Λ term, reduce to

$$\dot{H} + 3H^2 + \frac{2k}{a^2} = 8\pi V + \Lambda, \tag{52}$$

$$-3(\dot{H} + H^2) = 8\pi(\dot{\phi}^2 - V) - \Lambda, \tag{53}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 \tag{54}$$

Any one of equations (52)-(54) may be derived from the other two. Thus, we have a system of two independent equations in three unknowns a , ϕ and V .

In order to obtain a solution, an additional assumption is needed. For example [22] and [23] assumed forms of $V = V(\phi)$, [24] assumed forms of $a = a(t)$, and [25] assumed forms of $\phi = \phi(t)$.

Rather than using an arbitrary assumption, optimal control has successfully been used [26] to construct a model of a slow-roll inflationary universe with minimum change of the scalar field.

A slow-roll inflationary universe [26]

A slow-roll regime requires a slow change in the field ϕ . Thus, the evolution of the universe must be such that $\Delta\phi := \phi(t_2) - \phi(t_1)$ on any finite time interval $[t_1, t_2]$ would be the smallest among all other possible evolutions.

By adding equations (52) and (53), one obtains

$$\dot{\phi} = \sqrt{\frac{1}{4\pi}} \sqrt{\frac{k}{a^2} - \dot{H}}, \tag{55}$$

and then

$$\Delta\phi = \int_{t_1}^{t_2} \dot{\phi} dt = \sqrt{\frac{1}{4\pi}} \int_{t_1}^{t_2} \sqrt{\frac{k}{a^2} - \dot{H}} dt. \tag{56}$$

Taking a and H as the state functions, and $u = \dot{H}$ as the control function the optimal control problem is then formulated as follows. Determine the control function u which minimizes the objective function

$$J = \int_{t_1}^{t_2} \sqrt{\frac{k}{a^2} - u} dt, \quad (57)$$

subject to

$$\dot{a} = aH, \quad a(t_1) = a_1 = \text{const.}, \quad (58)$$

$$\dot{H} = u, \quad H(t_1) = H_1 = \text{const.} \quad (59)$$

It should be noted that, given a solution a, H, u to the above problem, the scalar field ϕ is determined by the integration of equation (55), and the potential V is determined as an output of the system by

$$8\pi V = \frac{2k}{a^2} + 3H^2 + u - \Lambda, \quad (60)$$

Application of Pontryagin's maximum principle leads to

$$2a^4(k - a^2u)\ddot{u} + 3a^6\dot{u}^2 + 12ka^2\dot{a}^2u + 4k^2a\ddot{a} + 4ka^3(3\dot{a}\dot{u} - u\ddot{a}) + 8k^3 + 8ka^4u^2 - 16k^2a^2u = 0. \quad (61)$$

For a flat universe, $k = 0$, the optimal solution is given by

$$u(t) = \frac{-1}{(a_0t + b_0)^2}, \quad a_0, b_0 \text{ constants}, \quad (62)$$

$$H(t) = \frac{1}{a_0} \frac{1}{(a_0t + b_0)} + c_0, \quad (63)$$

$$a(t) = a_2 (a_0t + b_0)^{\frac{1}{a_0^2}} e^{c_0t}, \quad (64)$$

where

$$c_0 = H_1 - [a_0(a_0t_1 + b_0)]^{-1}, \quad a_2 = a_1(a_0t_1 + b_0)^{-\frac{1}{a_0^2}} e^{-c_0t_1}.$$

Then one obtains

$$\phi(t) = \sqrt{\frac{1}{4\pi}} \frac{1}{a_0} \ln(a_0t + b_0) + \phi_0, \quad \phi_0 = \text{const.} \quad (65)$$

Finally, equations (62) and (60) give the potential

$$V(t) = \frac{1}{8\pi} \left[-\Lambda + 3c_0^2 + \frac{-1}{(a_0 t + b_0)^2} \left(\frac{3}{a_0^2} - 1 \right) + \frac{6}{a_0} \left(\frac{c_0}{a_0 t + b_0} \right) \right]. \quad (66)$$

The potential V , as a function of ϕ , is then given by

$$V(\phi) = \frac{1}{8\pi} \left[-\Lambda + 3c_0^2 + \left(\frac{3}{a_0^2} - 1 \right) e^{-a_0 \sqrt{16\pi}(\phi - \phi_0)} + \frac{6}{a_0} c_0 e^{-a_0 \sqrt{4\pi}(\phi - \phi_0)} \right]. \quad (67)$$

This solution with different constants, has been obtained in [27] using classical variational calculus.

6. Conclusion

We have given a review of four example problems in relativistic astrophysics and cosmology that have been formulated and solved as optimal control problems.

The first example in section 3 gives a proof of a variational principle that is brief and straightforward, compared to proofs using classical variational calculus. This advantage should be expected. Pontryagin's maximum principle already used variational calculus to derive the optimality conditions for a general problem form. Once the problem at hand is cast into that form, the optimality conditions can directly be applied. Variational principles are common in branches of physics, and could be candidates for optimal control application.

The second example in section 3 determines the configuration of a neutron star with maximum mass. The clever formulation of this problem yielded a typical optimal control problem that had been solved using optimal control methods.

For stellar models, Fujisawa et al [28] established a bound on the maximum M/R ratio, where a numerical solution was given. The problem was initially modeled as an optimal control problem. However, it was solved using variational calculus that required long derivations. It would be interesting to solve that problem using optimal control methods.

The other two examples are different; no objective function is initially given. However, they show that some of Einstein's field equations can, indeed, be modeled as optimal control models. Rather than using adhoc assumptions, an objective function is sought to be optimal. Then, the well-established techniques of optimal control would result in the optimal solutions.

This approach may be extended to other models with different configurations such as rotating stars, binary systems, and inhomogeneous and/or rotating universes. It may also be

extended to models with more realistic matter having, for example, anisotropic pressure and/or viscosity.

However, in constructing optimal control models, specifying state and control functions is not a simple task, and should take utmost care. Issues of controllability and existence may sometimes be handled. A more serious challenge is to specify some optimality criteria that should be physically relevant and should also yield a mathematically solvable problem. It would take a lot of hard work but could, in the end, lead to interesting and powerful results.

It is worth noting that a different application of optimal control to GR has recently been published. Ansel [29] presented a path integral formalism, based on the framework of optimal control theory, with a new Lagrangian different from the Einstein-Hilbert Lagrangian [30]. Einstein's field equations are then recovered exactly with variations of the new action functional.

- **Conflict of Interest**

The authors declare that they have no conflict of interest.

7. References

- [1] Pontryagin L.S., Boltyanskii V.G., Gamkrelidze R.V. and Mishchenko E.F. *The Mathematical Theory of Control Processes*. New York: Interscience, 1962.
- [2] Liberzon D. *Calculus of Variations and Optimal Control Theory, A Concise Introduction*. Princeton: Princeton University Press, 2012. [ISBN:9780691151878](https://doi.org/10.1017/9780691151878)
- [3] Bechhoefer J. *Control Theory for Physicists*. Cambridge: Cambridge University Press, 2021. <https://doi.org/10.1017/9780511734809>
- [4] Misner C.W., Thorne K.S. and Wheeler J.A. *Gravitation*. San Francisco: Freeman, 1973.
- [5] Harrison B.K., Thorne K.S., Wakano M. and Wheeler J.A. *Gravitation Theory and Gravitational Collapse*. Chicago: University of Chicago Press, 1965, p.16-18.
- [6] Weinberg S. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*. New York: John Wiley, 1972, p.306-307.
- [7] Tolman R.C. *Static Solutions of Einstein's Field Equations for Spheres of Fluid*. Phys. Rev., 1939, **55**, 364. DOI: <https://doi.org/10.1103/PhysRev.55.364>
- [8] Oppenheimer J.R. and Volkoff G. *On Massive Neutron Cores*. Phys. Rev., 1939, **55**, 374. DOI: <https://doi.org/10.1103/PhysRev.55.374>

- [9] Haggag S and Safko J. *The Condition of Hydrostatic Equilibrium of Stellar Models Using Optimal Control*. Astrophysics and Space Science, 2003, **283**, 369.
DOI: <https://doi.org/10.1023/A:1021684407650>
- [10] Rhoades C.E. and Ruffini R. *Maximum Mass of a Neutron Star*. Phys. Rev. Lett., 1974, **32**, 324. DOI: <https://doi.org/10.1103/PhysRevLett.32.324>
- [11] Ozer M. and Taha M.O. *A Possible Solution to the Main Cosmological Problems*. Phys. Lett., 1986, B **171**, 4. DOI: [https://doi.org/10.1016/03702693\(86\)91421-8](https://doi.org/10.1016/03702693(86)91421-8)
- [12] Ozer M. and Taha M.O. *A Model of the Universe Free of Cosmological Problems*. Nuclear Phys., 1987, B **287**, 776. DOI: [https://doi.org/10.1016/05503213\(87\)90128-3](https://doi.org/10.1016/05503213(87)90128-3)
- [13] Chen W. and Wu Y. *Implications of a Cosmological Constant Varying as R^{-2}* . Phys. Rev. D, 1990, **41**, 695. DOI: <https://doi.org/10.1103/PhysRevD.41.695>
- [14] Berman M. *Cosmological Models with a Variable Cosmological Term*. Phys. Rev. D, 1991, **43**, 1075. DOI: <https://doi.org/10.1103/PhysRevD.43.1075>
- [15] Al-Rawaf A.S and Taha M.O. *Cosmology of General Relativity without Energy-Momentum Conservation*. Gen. Rel. Grav., 1996, **28**, 935.
DOI: <https://doi.org/10.1007/BF02113090>
- [16] Al-Rawaf A.S. *A Cosmological Model with a Generalised Cosmological Constant*. Mod. Phys. Lett., 1998, A **13**, 29. DOI: <https://doi.org/10.1142/S0217732398000498>
- [17] Overduin J.M. and Cooperstock F.I. *Evolution of the Scale Factor with a Variable Cosmological Term*. Phys. Rev., 1998, D **58**, 043506.
DOI: <https://doi.org/10.1103/PhysRevD.58.043506>
- [18] Arbab A.I. *Cosmic Acceleration with a Positive Cosmological Constant*. Class. Quantum Gravity, 2003, **28**, 93. DOI: <https://doi.org/10.1088/0264-9381/20/1/307>
- [19] Arbab A.I. *Cosmological Consequences of a Built-In Cosmological Constant Model*. J. Cosmol. Astro. Phys., 2003, **05**, 008.
DOI: <https://doi.org/10.1088/1475-7516/2003/05/008>
- [20] Haggag S., Abdel-Hafeez S.A. and Ramadan M. *A Closed Universe with Maximum Lifetime Using Optimal Control*. 2018. <http://arxiv.org/abs/1905.11525>
- [21] Guth A.H. *The Inflationary Universe: a Possible Solution to the Horizon and Flatness Problems*. Phys. Rev. D, 1981, **23**, 347. DOI: <https://doi.org/10.1103/PhysRevD.23.347>

- [22] Barrow J.D. and Graham A.H. *Singular Inflation*. Phys. Rev. D, 2015, **91**, 083513. DOI: <https://doi.org/10.1103/PhysRevD.91.083513>
- [23] Parsons P. and Barrow J.D. *Generalised Scalar Field Potentials and Inflation*. Phys. Rev. D, 1995, **51**, 6757. DOI: <https://doi.org/10.1103/PhysRevD.51.6757>
- [24] Chervon S.V., Zhuravlev V.M. and Shchigolev V.K. *New Exact Solutions in Standard Inflationary Models*. Phys. Lett., 1997, **398**, 269. DOI: [https://doi.org/10.1016/S0370-2693\(97\)00238-4](https://doi.org/10.1016/S0370-2693(97)00238-4)
- [25] Barrow J.D. *Exact Inflationary Universes with Potential Minima*. Phys. Rev., 1994, **49**, 3055. DOI: <https://doi.org/10.1103/PhysRevD.49.3055>
- [26] Haggag S., Desokey F. and Ramadan M. *A Cosmological Inflationary Model Using Optimal Control*. Grav. and Cosmol., 2017, **23**, 236. DOI: <https://doi.org/10.1134/S0202289317030069>
- [27] Zhuravlev V.M., Chervon S.V. and Shchigolev V.K. *New Classes of Exact Solutions in Inflationary Cosmology* J. of Experim. and Theor. Phys., 1998, **87**, 223. DOI: <https://doi.org/10.1134/1.558649>
- [28] Fujisawa A., Saida H., Yoo Ch. and Nambu Y. *Maximum Mass of a Barotropic Spherical Star*. Class. Quantum Grav., 2015, **32**, 13. DOI: <https://doi.org/10.1088/0264-9381/32/21/215028>
- [29] Ansel Q. *Loop Quantum Gravity with Optimal Control Path Integral, and Application to Black Hole Tunneling*. Gen. Rel. Grav., 2022, **54**, 42. DOI: <https://doi.org/10.1007/s10714-022-02923-6>
- [30] Grøn, Ø. *Introduction to Einstein's Theory of Relativity*. Second Edition. Switzerland: Springer Nature, 2020, Section 7.6. <https://doi.org/10.1007/978-3-030-43862-3>