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Original Article

Recurrence Relations for Moment Generating Function of Progressive First Failure Censoring and Characterizations of Right Truncated Exponential Distribution

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ABSTRACT

In this article, we establish recurrence relations (RR) for moment generating function (MGF) based on progressive first failure censoring (PFFC). Characterization for right truncated exponential distribution (RTED) using relation between probability density function (pdf) and distribution function (cdf) and using RR of MGF of PFFC are also obtained. Further, the results are specialized to the progressively type-II right censored order statistics (PTIIRCOS).

Keywords: Characterization; moment generating function; progressive first failure censoring; recurrence relations; right truncated exponential distribution.

1. Introduction

PFFC can be described as follows, suppose a lifetime test is administered to n separate groups with k items in each group. R_1 groups and the group in which the first failure is observed are randomly removed from the test when the first failure $X_{1:m:n,k}^{(R_1,R_2,R_3,\dots,R_{m-1},R_m)}$ has occurred, R_2 groups with the group in which the 2nd failure is observed are randomly removed from the test at $X_{2:m:n,k}^{(R_1,R_2,R_3,\dots,R_{m-1},R_m)}$, and so on until the m^{th} failure $X_{m:m:n,k}^{(R_1,R_2,R_3,\dots,R_{m-1},R_m)}$ is observed, the remaining groups R_m are removed from the test. Then $X_{1:m:n,k}^{(R_1,R_2,R_3,\dots,R_{m-1},R_m)} < X_{2:m:n,k}^{(R_1,R_2,R_3,\dots,R_{m-1},R_m)} < \cdots < X_{m:m:n,k}^{(R_1,R_2,R_3,\dots,R_{m-1},R_m)}$ are called PFFC order statistics with the progressive censored scheme $(R_1, R_2, R_3, \dots, R_{m-1}, R_m)$, where $n = m + \sum_{i=1}^{m} R_i$. If the failure times of the $n \times k$ items originally in the test are from a continuous population with pdf and cdf, the joint pdf for $X_{1:m:n,C}^{(R_1,R_2,R_3,\ldots,R_{m-1},R_m)}, \ldots, X_{m:m:n,C}^{(R_1,R_2,R_3,\ldots,R_{m-1},R_m)}$ is defined as follows:

$$f_{X_{1:m:n},\ldots,X_{m:m:n}}(x_1, x_2, \ldots, x_m) = K_{(n,m-1)} C^m \prod_{i=1}^{m} f(x_i, \theta)[\bar{F}(x_i, \theta)]^{M_i},$$

$$0 < x_1 < x_2 << x_m < \infty, \tag{1.1}$$

where,

$K_{(n,m-1)} = n(n - R_1 - 1) \ldots (n - R_1 - R_2 \ldots - R_{m-1} - m + 1)$, and $M_i = cR_i + c - 1$.

Marwa Mohie El-Din and Sharawy [6] derived characterization for generalized exponential distribution. Kotb et al. [5] derived E-Bayesian estimation for Kumaraswamy distribution using PFFC. Abu-Moussa et al. [1] derived Estimation of Reliability Functions for the Extended Rayleigh Distribution via PFFC see more Sharawy [2,3].

For any continuous distribution, we denote the single MGF of the PFFC in view of Eq. (1.1) as

$$M_{q:m:n,C}^{(M_1,M_2,\ldots,M_m)} = E\left[e^{tx_q} X_{q:m:n,C}^{(M_1,M_2,\ldots,M_m)}\right] =$$

$$K_{(n,m-1)} \iint \cdots \int_{0<x_1<\cdots<x_m<\infty} e^{tx_q} C^m f(x_1)[\bar{F}(x_1)]^{M_1} \times$$

$$f(x_2)[\bar{F}(x_2)]^{M_2} \ldots f(x_m)[\bar{F}(x_m)]^{M_m} dx_1 \ldots dx_m, \tag{1.2}$$

and product MGF

$$M_{q,s:m:n,C}^{(M_1,M_2,\ldots,M_m)} = K_{(n,m-1)} \iint \cdots \int_{0<x_1<\cdots<x_m<\infty} e^{(t_1 x_q + t_2 x_s)} C^m \times$$

$$f(x_1)[\bar{F}(x_1)]^{M_1} \ldots f(x_m)[\bar{F}(x_m)]^{M_m} dx_1 \ldots dx_m. \tag{1.3}$$

2. Recurrence Relations

In this section we introduce the RR for the MGF for RTED based on PFFC.

The pdf of RTED is

$$f(x) = \frac{\theta}{P} e^{-\theta x}, \quad 0 \le x \le P_1, \quad where \quad P_1 = -\ln(1 - \mathrm{P}). \tag{2.1}$$

Here, $1 - P$ is the proportion of right truncation of the standard exponential distribution.

The cdf of RTED is given by

$$F(x) = \frac{1}{P} - \frac{1}{P} e^{-\theta x}. \tag{2.2}$$

The RTED is used to simulate the lifespans of wear-and-tear components.

The relation between pdf and cdf is given by,

$$f(x) = \frac{\theta}{P} - \theta + \theta[\bar{F}(x)]. \qquad (2.3)$$

In the next theorem we introduce the RR for MGF based on PFFC from a RTED.

2.1 Recurrence relation of single moment generating function of progressive first failure censoring

In the next theorem we introduce the recurrence relation for single MGF of PFFC.

Theorem 2.1

For $2 \leq q \leq m-1$ and $m \leq n$ then

$$M_{q:m:n,C}^{(M_1,M_2,\dots,M_m)}\left[\frac{t}{\theta} - \left(M_q+1\right)\right] = \left(\frac{1}{P}-1\right)\frac{M_q\ K_{(n,q-1)}}{K_{(n-1,q-1)}} M_{q:m:n-1,C}^{\left(M_1,\dots,M_{q-1},M_q-1,M_{q+1},\dots,M_m\right)}$$

$$+\left(\frac{1}{P}-1\right)\frac{K_{(n,q)}}{K_{(n-1,q-1)}} M_{q:m-1:n-1,C}^{\left(M_1,\dots,M_{q-1},M_q+M_{q+1},M_{q+2},\dots,M_m\right)}$$

$$-\left(\frac{1}{P}-1\right)\frac{K_{(n,q-1)}}{K_{(n-1,q-2)}} M_{q-1:m-1:n-1,C}^{\left(M_1,\dots,M_{q-1}+M_q,M_{q+1},\dots,M_m\right)}$$

$$-\left(n-R_1-\cdots-R_{q-1}-q+1\right)\left[M_{q-1:m-1:n,C}^{\left(M_1,\dots,M_{q-1}+M_q+1,M_{q+1},\dots,M_m\right)}\right]$$

$$+\left(n-R_1-\cdots-R_q-q\right)\left[M_{q:m-1:n,C}^{\left(M_1,\dots,M_{q-1},M_q+M_{q+1}+1,M_{q+2},\dots,M_m\right)}\right]. \qquad (2.4)$$

Proof

From Eq. (1.2) and Eq. (2.3), we get

$$M_{q:m:n,C}^{(M_1,M_2,\dots,M_m)} = K_{(n,m-1)} \iint \cdots \int_{0<x_1<\cdots<x_{q-1}<x_{q+1}<\cdots<x_m<\infty} I_1\left(x_{q-1},x_{q+1}\right)C^m \times$$

$$f(x_1)[\bar{F}(x_1)]^{M_1} \dots f\left(x_{q-1}\right)\left[\bar{F}\left(x_{q-1}\right)\right]^{M_{q-1}} f\left(x_{q+1}\right) \times$$

$$\left[\bar{F}\left(x_{q+1}\right)\right]^{M_{q+1}} \dots f(x_m)[\bar{F}(x_m)]^{M_m} dx_1 \dots dx_{q-1} dx_{q+1} \dots dx_m, \qquad (2.5)$$

where

$$I_1\left(x_{q-1},x_{q+1}\right) = \theta \int_{x_{q-1}}^{x_{q+1}} \left[\left(\frac{1}{P}-1\right) + \left[\bar{F}\left(x_q\right)\right]\right] e^{tx_q} \left[\bar{F}\left(x_q\right)\right]^{M_q} dx_q. \qquad (2.6)$$

Now, integrating by parts gives

$$I_1(x_{q-1},x_{q+1})=\frac{\theta}{t}\left(\frac{1}{P}-1\right)\left[e^{tx_{q+1}}\left[\bar{F}(x_{q+1})\right]^{M_q}-e^{tx_{q-1}}\left[\bar{F}(x_{q-1})\right]^{M_q}\right]$$

$$+\frac{\theta(M_q)}{t}\left(\frac{1}{P}-1\right)\int_{x_{q-1}}^{x_{q+1}}e^{tx_q}f(x_q)\left[\bar{F}(x_q)\right]^{M_q-1}dx_q$$

$$+\frac{\theta}{t}\left[e^{tx_{q+1}}\left[\bar{F}(x_{q+1})\right]^{M_q+1}-e^{tx_{q-1}}\left[\bar{F}(x_{q-1})\right]^{M_q+1}\right]$$

$$+\frac{\theta(M_q+1)}{t}\int_{x_{q-1}}^{x_{q+1}}e^{tx_q}f(x_q)\left[\bar{F}(x_q)\right]^{M_q}dx_q. \tag{2.7}$$

Substituting by Eq. (2.7) in Eq. (2.5) and simplifying, yields Eq. (2.4).

This completes the proof.

Special case

Theorem 2.1 will be valid for the PTIIRCOS as a special case from the PFFC when $C=1$,

$$M_{q:m:n}^{(R_1,\dots,R_m)}\left[\frac{t}{\theta}-(R_q+1)\right]=\left(\frac{1}{P}-1\right)\frac{R_q\,K_{(n,q-1)}}{K_{(n-1,q-1)}}M_{q:m:n-1}^{(R_1,\dots,R_{q-1},R_q-1,R_{q+1},\dots,R_m)}$$

$$+\left(\frac{1}{P}-1\right)\frac{K_{(n,q)}}{K_{(n-1,q-1)}}M_{q:m-1:n-1}^{(R_1,\dots,R_{q-1},R_q+R_{q+1},R_{q+2},\dots,R_m)}$$

$$-\left(\frac{1}{P}-1\right)\frac{K_{(n,q-1)}}{K_{(n-1,q-2)}}M_{q-1:m-1:n-1}^{(R_1,\dots,R_{q-2},R_{q-1}+R_q,R_{q+1},\dots,R_m)}$$

$$-(n-R_1-\cdots-R_{q-1}-q+1)\left[M_{q-1:m-1:n}^{(R_1,\dots,R_{q-2},R_{q-1}+R_q+1,R_{q+1},\dots,R_m)}\right]$$

$$+(n-R_1-\cdots-R_q-q)\left[M_{q:m-1:n}^{(R_1,\dots,R_{q-1},R_q+R_{q+1}+1,R_{q+2},\dots,R_m)}\right].$$

2.2 Recurrence relation of product moment generating function of progressive first failure censoring

In the next two theorems we introduce the RR for product MGF based on PFFC from RTED.

Theorem 2.2

For $1\le q<s\le m-1$ and $m\le n$, then

$$M_{q,s:m:n,C}^{(M_1,M_2,\dots,M_m)}\left[\frac{t_1}{\theta}-(M_q+1)\right]=\left(\frac{1}{P}-1\right)\frac{M_q\,K_{(n,q-1)}}{K_{(n-1,q-1)}}M_{q,s:m:n-1,C}^{(M_1,\dots,M_{q-1},M_q-1,M_{q+1},\dots,M_m)}$$

$$+\left(\frac{1}{P}-1\right)\frac{K_{(n,q)}}{K_{(n-1,q-1)}}M_{q,s-1:m-1:n-1,C}^{(M_1,\dots,M_{q-1},M_q+M_{q+1},M_{q+2},\dots,M_m)}$$

$$-\left(\frac{1}{P}-1\right)\frac{K_{(n,q-1)}}{K_{(n-1,q-2)}}M_{q-1,s-1:m-1:n-1,C}^{(M_1,\ldots,M_{q-1}+M_q,M_{q+1},\ldots,M_m)}$$

$$-(n-R_1-\cdots-R_{q-1}-q+1)\left[M_{q-1,s-1:m-1:n}^{(M_1,\ldots,M_{q-1}+M_q+1,M_{q+1},\ldots,M_m)}\right]$$

$$+(n-R_1-\cdots-R_q-q)\left[M_{q,s-1:m-1:n}^{(M_1,\ldots,M_{q-1},M_q+M_{q+1}+1,M_{q+2},\ldots,M_m)}\right]. \quad (2.8)$$

Proof

Similarly as proved in theorem 2.1.

Theorem 2.3

For $1 \le q < s \le m-1$ and $m \le n$, then

$$M_{q,s:m:n,C}^{(M_1,M_2,\ldots,M_m)}\left[\frac{t_2}{\theta}-(M_s+1)\right]=\left(\frac{1}{P}-1\right)\frac{M_s\,K_{(n,s-1)}}{K_{(n-1,s-1)}}M_{q,s:m:n-1,C}^{(M_1,\ldots,M_{s-1},M_s-1,M_{s+1},\ldots,M_m)}$$

$$+\left(\frac{1}{P}-1\right)\frac{K_{(n,s)}}{K_{(n-1,s-1)}}M_{q,s-1:m-1:n-1}^{(M_1,\ldots,M_{s-1},M_s+M_{s+1},M_{s+2},\ldots,M_m)}$$

$$-\left(\frac{1}{P}-1\right)\frac{K_{(n,s-1)}}{K_{(n-1,s-2)}}M_{q-1,s-1:m-1:n-1}^{(M_1,\ldots,M_{s-1}+M_s,M_{s+1},\ldots,M_m)}$$

$$-(n-R_1-\cdots-R_{s-1}-s+1)\left[M_{q,s-1:m-1:n}^{(M_1,\ldots,M_{s-1}+M_s+1,M_{s+1},\ldots,M_m)}\right]$$

$$+(n-R_1-\cdots-R_s-s)\left[M_{q,s:m-1:n}^{(M_1,\ldots,M_{s-1},M_q+M_{s+1}+1,M_{s+2},\ldots,M_m)}\right]. \quad (2.9)$$

Proof

Similarly as proved in theorem 2.1.

3. The Characterizations of the right truncated exponential distribution via differential equation and recurrence relation for single and product moment generating function

In this section we introduce the characterization of the RTED.

3.1 Characterization via differential equation for the right truncated exponential distribution

In the next theorem we introduce the characterization of the RTED.

Theorem 3.1

Let X be a continuous random variable. Then X has right truncated exponential distribution iff (2.1) holds

Proof

Necessity:

From Eq. (2.1) and Eq. (2.2) we can easily obtain Eq. (2.3).

Sufficiency:

Suppose that Eq. (2.3) is true. Then we have:

$$dF(x) + \theta F(x) = \frac{\theta}{P}. \tag{3.1}$$

Now, (3.2) is a linear first order nonhomogeneous differential equation in the unknown function (x). We first solve the equation

$$dF(x) + \theta F(x) = 0. \tag{3.2}$$

Separation of variables gives the general solution of this equation as:

$$F(x) = Be^{-\theta x}\,, \tag{3.3}$$

where B is an arbitrary constant.

Now, to get the general solution of (3.1) we use the method of variation of constants (Stepanov [7]).

Thus, by considering the constant B in (2.3) as a function of x, say, B(x); we have

$$F(x) = B(x)e^{-\theta x}\,. \tag{3.4}$$

Consequently,

$$dF(x) = -\theta B(x)e^{-\theta x} + e^{-\theta x}dB(x). \tag{3.5}$$

Therefore, substituting (3.4) and (3.5) in (3.1) and simplifying we get

$$dB(x) = \frac{\theta}{P}\ e^{\theta x}.$$

On integrating, we get

$$B(x) = \frac{e^{\theta x}}{P} + B_1, \tag{3.6}$$

where B_1 is an arbitrary constant.

From (3.4) and (3.6), the complete solution of (3.1) is given by

$$F(x) = \left[\frac{e^{\theta x}}{P} + B_1\right]e^{-\theta x}. \tag{3.7}$$

Now, since $F(0) = 0$, then putting $x = 0$ in this equation, we get $B_1 = \frac{-1}{P}$

Therefore, $F(x) = \frac{1}{P} - \frac{1}{P}e^{-\theta x}$.

That is the distribution function of RTED.

This completes the proof.

3.2 Characterization via recurrence relations for MGF

In the next theorem we introduce the characterization of the RTED using MGF of PFFC.

Theorem 3.2

Let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be the order stasistics of a random sample of size n. Then X has RTED iff (2.4) holds, for $2 \leq q \leq m-1$ and $m \leq n$

Necessity:

Theorem 2.1 proves the necessary part of this theorem.

Sufficiency:

Assuming that (3.4) holds, then we have:

$$M_{q:m:n,C}^{(M_1,M_2,\ldots,M_m)}\left[\frac{t}{\theta} - \left(M_q + 1\right)\right] = \left(\frac{1}{P} - 1\right)\frac{M_q\ K_{(n,q-1)}}{K_{(n-1,q-1)}} M_{q:m:n-1,C}^{(M_1,\ldots,M_{q-1},M_q-1,M_{q+1},\ldots,M_m)}$$

$$+\left(\frac{1}{P} - 1\right)\frac{K_{(n,q)}}{K_{(n-1,q-1)}} M_{q:m-1:n-1,C}^{(M_1,\ldots,M_{q-1},M_q+M_{q+1},M_{q+2},\ldots,M_m)}$$

$$-\left(\frac{1}{P} - 1\right)\frac{K_{(n,q-1)}}{K_{(n-1,q-2)}} M_{q-1:m-1:n-1,C}^{(M_1,\ldots,M_{q-1}+M_q,M_{q+1},\ldots,M_m)}$$

$$-\left(n - R_1 - \cdots - R_{q-1} - q + 1\right)\left[M_{q-1:m-1:n,C}^{(M_1,\ldots,M_{q-1}+M_q+1,M_{q+1},\ldots,M_m)}\right]$$

$$+\left(n - R_1 - \cdots - R_q - q\right)\left[M_{q:m-1:n,C}^{(M_1,\ldots,M_{q-1},M_q+M_{q+1}+1,M_{q+2},\ldots,M_m)}\right]. \quad (3.8)$$

where,

$$M_{q:m:n,k}^{(M_1,M_2,\ldots,M_m)} = K_{(n,m-1)} \iint \cdots \int_{0<x_1<\cdots<x_{q-1}<x_{q+1}<\cdots<x_m<\infty} U_1\left(x_{q-1}, x_{q+1}\right)C^m \times$$
$$f(x_1)[\bar{F}(x_1)]^{M_1} \ldots f\left(x_{q-1}\right)\left[\bar{F}\left(x_{q-1}\right)\right]^{M_{q-1}} f\left(x_{q+1}\right)\left[\bar{F}\left(x_{q+1}\right)\right]^{M_{q+1}} \times$$
$$\ldots f(x_m)[\bar{F}(x_m)]^{M_m} dx_1 \ldots dx_{q-1} dx_{q+1} \ldots dx_m, \quad (3.9)$$

and

$$U_1\left(x_{q-1}, x_{q+1}\right) = \int_{x_{q-1}}^{x_{q+1}} e^{tx_q} f\left(x_q\right)\left[\bar{F}\left(x_q\right)\right]^{M_q} dx_q. \tag{3.10}$$

Now, integrating by parts gives

$$U_1\left(x_{q-1}, x_{q+1}\right) = \frac{-te^{tq+1}\left[\bar{F}\left(x_{q+1}\right)\right]^{M_q+1} + te^{tx_{q-1}}\left[\bar{F}\left(x_{q-1}\right)\right]^{M_q+1}}{M_q+1} + \frac{t}{M_q+1}\int_{x_{q-1}}^{x_{q+1}} e^{tx_q}\left[\bar{F}\left(x_q\right)\right]^{M_q+1} dx_q, \tag{3.11}$$

similarly

$$M_{q:m:n-1,C}^{(M_1,\ldots,M_{q-1},M_q-1,M_{q+1},\ldots,M_m)} = K_{(n,m-1)} \iint \cdots \int_{0<x_1<\cdots<x_{q-1}<x_{q+1}<\cdots<x_m<\infty} U_2\left(x_{q-1}, x_{q+1}\right) C^m \times f(x_1)[\bar{F}(x_1)]^{M_1} \ldots f\left(x_{q-1}\right)\left[\bar{F}\left(x_{q-1}\right)\right]^{M_{q-1}} f\left(x_{q+1}\right)\left[\bar{F}\left(x_{q+1}\right)\right]^{M_{q+1}} \times \ldots f(x_m)[\bar{F}(x_m)]^{M_m} dx_1 \ldots dx_{q-1} dx_{q+1} \ldots dx_m, \tag{3.12}$$

where

$$U_2\left(x_{q-1}, x_{q+1}\right) = \frac{-te^{tq+1}\left[\bar{F}\left(x_{q+1}\right)\right]^{M_q} + te^{tx_{q-1}}\left[\bar{F}\left(x_{q-1}\right)\right]^{M_q}}{M_q} + \frac{t}{M_q}\int_{x_{q-1}}^{x_{q+1}} e^{tx_q}\left[\bar{F}\left(x_q\right)\right]^{M_q} dx_q. \tag{3.13}$$

Now by substituting for $M_{q:m:n,k}^{(M_1,M_2,\ldots,M_m)}$ and $M_{q:m:n-1,C}^{(M_1,\ldots,M_{q-1},M_q-1,M_{q+1},\ldots,M_m)}$ from (3.9) and (3.12) in (3.8), we get

$$K_{(n,m-1)} \iint \cdots \int_{0<x_1<\cdots<x_{q-1}<x_{q+1}<\cdots<x_m<\infty} e^{tx_q} C^m f\left(x_{q-1}\right)\left[\bar{F}\left(x_q\right)\right]^{M_q} f(x_1) \times [\bar{F}(x_1)]^{M_1} \ldots f\left(x_{q-1}\right)\left[\bar{F}\left(x_{q-1}\right)\right]^{M_{q-1}} f\left(x_{q+1}\right)\left[\bar{F}\left(x_{q+1}\right)\right]^{M_{q+1}} \times \ldots f(x_m)[\bar{F}(x_m)]^{M_m} dx_1 dx_2 \ldots dx_{q-1} dx_{q+1} \ldots dx_m =$$

$$\theta C_{(n,m-1)} \iint \cdots \int_{0<x_1<\cdots<x_{q-1}<x_{q+1}<\cdots<x_m<\infty} e^{tx_q} k^m \left[\bar{F}\left(x_q\right)\right]^{M_q+1} f(x_1) \times [\bar{F}(x_1)]^{M_1} \ldots f\left(x_{q-1}\right)\left[\bar{F}\left(x_{q-1}\right)\right]^{M_{q-1}} f\left(x_{q+1}\right)\left[\bar{F}\left(x_{q+1}\right)\right]^{M_{q+1}} \times \ldots f(x_m)[\bar{F}(x_m)]^{M_m} dx_1 dx_2 \ldots dx_{q-1} dx_{q+1} \ldots dx_m$$

$$(\frac{\theta}{P}-\theta)C_{(n,m-1)}\iint\dots\int_{0<x_1<\cdots<x_{q-1}<x_{q+1}<\cdots<x_m<\infty} e^{tx_q}k^m\left[\bar{F}\left(x_q\right)\right]^{M_q} f(x_1)\times$$
$$[\bar{F}(x_1)]^{M_1}\dots f\left(x_{q-1}\right)\left[\bar{F}\left(x_{q-1}\right)\right]^{M_{q-1}} f\left(x_{q+1}\right)\left[\bar{F}\left(x_{q+1}\right)\right]^{M_{q+1}}\times$$
$$\dots f(x_m)[\bar{F}(x_m)]^{M_m}dx_1dx_2\dots dx_{q-1}dx_{q+1}\dots dx_m.$$

We get

$$K_{(n,m-1)}\iint\dots\int_{0<x_1<\cdots<x_{q-1}<x_{q+1}<\cdots<x_m<\infty} e^{tx_q}C^m\left[f\left(x_q\right)-\frac{\theta}{P}+\theta-\theta\left[\bar{F}\left(x_q\right)\right]\right]\times$$
$$\left[\bar{F}\left(x_q\right)\right]^{M_q} f(x_1)[\bar{F}(x_1)]^{M_1}\dots f\left(x_{q-1}\right)\left[\bar{F}\left(x_{q-1}\right)\right]^{M_{q-1}}\times$$
$$f\left(x_{q+1}\right)\left[\bar{F}\left(x_{q+1}\right)\right]^{M_{q+1}}\dots f(x_m)[\bar{F}(x_m)]^{M_m}\times$$
$$dx_1dx_2\dots dx_{q-1}dx_{q+1}\dots dx_m=0. \quad (3.14)$$

Using Muntz-Szasz theorem, [See, Hwang and Lin [4]], I get

$$f(x)=\frac{\theta}{P}-\theta+\theta[\bar{F}(x)].$$

Using Theorem 3.1, I get

$$F(x)=\frac{1}{P}-\frac{1}{P}e^{-\theta x}.$$

That is the cdf of RTED.

This completes the proof.

In the next two theorems, we will introduce the characterization of theRTED using RR for product MGF based on PFFC.

Theorem 3.3

Let $X_{1:n}\le X_{2:n}\le\dots\le X_{n:n}$ be the order stasistics of a random sample of size n. Then X has RTED iff (2.8) holds, for $1\le q<s\le m-1$ and $m\le n$,

Proof

Similarly as proved in theorem 3.2.

Theorem 3.4

Let $X_{1:n}\le\dots\le X_{n:n}$ be the order stasistics of a random sample of size n. Then X has RTED iff (2.9) holds, for $1\le q<s\le m-1$, $m\le n$,

Proof

Similarly as proved in theorem 3.2.

- **Conflict of Interest**

The Authors declare no conflict of interest

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